UNIT 10 LARGE SAMPLE TESTS

Structure

10.1 INTRODUCTION

In previous unit, we have defined basic terms used in testing of hypothesis. After providing you necessary material required for any test, we can move towards discussing particular tests one by one. But before doing that let us tell you the strategy we are adopting here.

First we categories the tests under two heads:

- Large sample tests
- Small sample tests

After that, their unit wise distribution is done. In this unit, we will discuss large sample tests whereas in Units 11 and 12 we will discuss small sample tests. The tests which are described in these units are known as **"parametric tests"**.

Sometimes in our studies in the fields of economics, psychology, medical, etc. we take a sample of objects / units / participants / patients, etc. such as 70, 500, 1000, 10,000, etc. This situation comes under the category of large samples.

As a thumb rule, a sample of size n is treated as a large sample only if it contains more than 30 units (or observations, $n > 30$). And we know that, for large sample ($n > 30$), one statistical fact is that almost all sampling distributions of the statistic(s) are closely approximated by the normal distribution. Therefore, the test statistic, which is a function of sample observations based on $n > 30$, could be assumed follow the normal distribution approximately (or exactly).

But story does not end here. There are some other issues which need to be taken care off. Some of these issues have been highlighted by making different cases in each test as you will see when go through Sections 10.3 to 10.8 of this unit.

This unit is divided into ten sections. Section 10.1 is introductory in nature. General procedure of testing of hypothesis for large samples is described in

Section 10.2. In Section 10.3, testing of hypothesis for population mean is discussed whereas in Section 10.4, testing of hypothesis for difference of two population means with examples is described. Similarly, in Sections 10.5 and 10.6, testing of hypothesis for population proportion and difference of two population proportions are explained respectively. Testing of hypothesis for population variance and two population variances are described in Sections 10.7 and 10.8 respectively. Unit ends by providing summary of what we have discussed in this unit in Section 10.9 and solution of exercises in Section 10.10.

Objectives

After studying this unit, you should be able to:

- judge for a given situation whether we should go for large sample test or not;
- Applying the Z-test for testing the hypothesis about the population mean and difference of two population means;
- Applying the Z-test for testing the hypothesis about the population proportion and difference of two population proportions; and
- Applying the Z-test for testing the hypothesis about the population variance and two population variances.

10.2 PROCEDURE OF TESTING OF HYPOTHESIS FOR LARGE SAMPLES

As we have described in previous section that for large sample size (n > 30), one statistical fact is that almost all sampling distributions of the statistic(s) are closely approximated by the normal distribution. Therefore, when sample size is large one can apply the normal distribution based test procedures to test the hypothesis.

In previous unit, we have given the procedure of testing of hypothesis in general. Let us now discuss the procedure of testing a hypothesis for large samples in particular.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n (> 30) selected from a population having unknown parameter θ and we want to test the hypothesis about the hypothetical / claimed / assumed value θ_0 of parameter θ. For this, a test procedure is required. We discuss it step by step as follows:

Step I: First of all, we have to setup null hypothesis H_0 and alternative hypothesis H_1. Here, we want to test the hypothetical / claimed / assumed value θ_0 of parameter θ. So we can take the null and alternative hypotheses as

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta \neq \theta_0 \qquad [\text{for two-tailed test}]$$

or

$$\left.\begin{array}{l} H_0 : \theta \le \theta_0 \text{ and } H_1 : \theta > \theta_0 \\ H_0 : \theta \ge \theta_0 \text{ and } H_1 : \theta < \theta_0 \end{array}\right\} \qquad [\text{for one-tailed test}]$$

In case of comparing same parameter of two populations of interest, say, θ_1 and θ_2, then our null and alternative hypotheses would be

$$H_0 : \theta_1 = \theta_2 \text{ and } H_1 : \theta_1 \neq \theta_2 \qquad [\text{for two-tailed test}]$$

$$\text{or} \quad \left.\begin{array}{l} H_0: \theta_1 \le \theta_2 \text{ and } H_1: \theta_1 > \theta_2 \\ H_0: \theta_1 \ge \theta_2 \text{ and } H_1: \theta_1 < \theta_2 \end{array}\right\} \quad [\text{for one-tailed test}]$$

Step II: After setting the null and alternative hypotheses, we have to choose level of significance. Generally, it is taken as 5% or 1% ($\alpha = 0.05$ or 0.01). And accordingly rejection and non-rejection regions will be decided.

Step III: Third step is to determine an appropriate test statistic, say, Z in case of large samples. Suppose T_n is the sample statistic such as sample mean, sample proportion, sample variance, etc. for the parameter θ then for testing the null hypothesis, test statistic is given by

$$Z = \frac{T_n - E(T_n)}{SE(T_n)} = \frac{T_n - E(T_n)}{\sqrt{Var(T_n)}} \quad \left[\begin{array}{l} \because \text{ we know that SE of a statistic is} \\ \text{the SD of the sampling distribution} \\ \text{of that statistic} \\ \therefore SE(T_n) = SD(T_n) = \sqrt{Var(T_n)} \end{array}\right]$$

where, $E(T_n)$ is the expectation (or mean) of T_n and $Var(T_n)$ is variance of T_n.

Step IV: As already mentioned for large samples, statistical fact is that almost all sampling distributions of the statistic(s) are closely approximated by the normal distribution as the parent population is normal or non-normal. So, the test statistic Z will assumed to be approximately normally distributed with mean 0 and variance 1 as

$$Z = \frac{T_n - E(T_n)}{\sqrt{Var(T_n)}} \sim N(0,1)$$

By putting the values of T_n, $E(T_n)$ and $Var(T_n)$ in above formula we calculate the value of test statistic Z. Let z be the calculated value of test statistic Z.

Step V: After that, we obtain the critical (cut-off or tabulated) value(s) in the sampling distribution of the test statistic Z corresponding to α assumed in Step II. These critical values are given in **Table-I (Z-table)** at the Appendix of Block 1 of this course corresponding to different level of significance (α). For convenient some useful critical values at $\alpha = 0.01, 0.05$ for Z-test are given in **Table 10.1** in this section. After that, we construct rejection (critical) region of size α in the probability curve of the sampling distribution of test statistic Z.

Step VI: Take the decision about the null hypothesis based on the calculated and critical values of test statistic obtained in Step IV and Step V. Since critical value depends upon the nature of the test that it is one-tailed test or two-tailed test so following cases arise:

In case of one-tailed test:

Case I: When $H_0: \theta \le \theta_0$ and $H_1: \theta > \theta_0$ (right-tailed test)

In this case, the rejection (critical) region falls under the right tail of the probability curve of the sampling distribution of test statistic Z. Suppose z_α is the critical value at α level of significance so entire region greater than or equal to z_α is the rejection region and less than z_α is the non-rejection region as shown in Fig. 10.1.

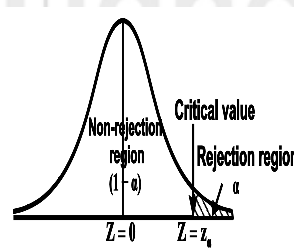

Fig. 10.1

If z (calculated value) $\geq z_\alpha$ (tabulated value), that means the calculated value of test statistic Z lies in the rejection region, then we reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that sample data provides us sufficient evidence against the null hypothesis and there is a significant difference between hypothesized or specified value and observed value of the parameter.

If $z < z_\alpha$, that means the calculated value of test statistic Z lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that the sample data fails to provide us sufficient evidence against the null hypothesis and the difference between hypothesized value and observed value of the parameter due to fluctuation of sample.

so the population parameter θ may be θ_0.

Case II: When $H_0 : \theta \geq \theta_0$ and $H_1 : \theta < \theta_0$ (left-tailed test)

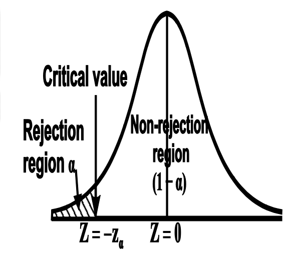

Fig. 10.2

In this case, the rejection (critical) region falls under the left tail of the probability curve of the sampling distribution of test statistic Z. Suppose $-z_\alpha$ is the critical value at α level of significance then entire region less than or equal to $-z_\alpha$ is the rejection region and greater than $-z_\alpha$ is the non-rejection region as shown in Fig. 10.2.

If $z \leq -z_\alpha$, that means the calculated value of test statistic Z lies in the rejection region, then we reject the null hypothesis H_0 at α level of significance.

If $z > -z_\alpha$, that means the calculated value of test statistic Z lies in the non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

In case of two-tailed test: When $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$

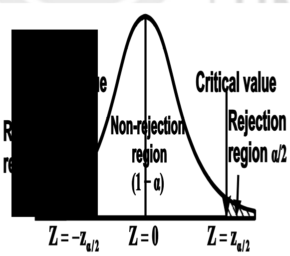

Fig. 10.3

In this case, the rejection region falls under both tails of the probability curve of sampling distribution of the test statistic Z. Half the area (α) i.e. $\alpha/2$ will lies under left tail and other half under the right tail. Suppose $-z_{\alpha/2}$ and $z_{\alpha/2}$ are the two critical values at the left-tailed and right-tailed respectively. Therefore, entire region less than or equal to $-z_{\alpha/2}$ and greater than or equal to $z_{\alpha/2}$ are the rejection regions and between $-z_{\alpha/2}$ and $z_{\alpha/2}$ is the non-rejection region as shown in Fig. 10.3.

If $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$, that means the calculated value of test statistic Z lies in the rejection region, then we reject the null hypothesis H_0 at α level of significance.

If $-z_{\alpha/2} < z < z_{\alpha/2}$, that means the calculated value of test statistic Z lies in the non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

Table 10.1 shows some commonly used critical (cut-off or tabulated) values for one-tailed test and two-tailed test at different level of significance (α) for Z-test.

Table 10.1: Critical Values for Z-test

Level of Significance (α)	Two-Tailed Test	One-Tailed Test	
		Right-Tailed Test	Left- Tailed Test
$\alpha = 0.05$ (= 5%)	$\pm z_{\alpha/2} = \pm 1.96$	$z_\alpha = 1.645$	$-z_\alpha = -1.645$
$\alpha = 0.01$ (= 1%)	$\pm z_{\alpha/2} = \pm 2.58$	$z_\alpha = 2.33$	$-z_\alpha = -2.33$

Note 1: As we have discussed in Step IV of this procedure that when sample size is large then test statistic follows the normal distribution as the parent population is normal or non-normal so we do not require any assumption of the form of the parent population for large sample size but when sample size is small (n < 30) then for applying parametric test we must require the assumption that the population is normal as we shall in Units 11 and 12. If this assumption is not fulfilled then we apply the non-parametric tests which will be discussed in Block 4 of this course.

Decision making procedure about the null hypothesis using the concept of p-value:

To take the decision about the null hypothesis on the basis of p-value, the p-value is compared with given level of significance (α) and if p-value is less than or equal to α then we reject the null hypothesis and if the p-value is greater than α we do not reject the null hypothesis.

Since test statistic Z follows approximately normal distribution with mean 0 and variance unity, i.e. standard normal distribution and we also know that standard normal distribution is symmetrical about Z = 0 line therefore, if z represents the calculated value of Z then p-value can be calculated as follows:

For one-tailed test:

For $H_1: \theta > \theta_0$ (right-tailed test)

$$\text{p-value} = P[Z \geq z]$$

For $H_1: \theta < \theta_0$ (left-tailed test)

$$\text{p-value} = P[Z \leq z]$$

For two-tailed test:

For $H_1: \theta \neq \theta_0$

$$\text{p-value} = 2P\left[Z \geq |z|\right]$$

These p-values for Z-test can be obtained with the help of **Table-I (Z-table)** given in the Appendix at the end of Block 1 of this course (which gives the probability $[0 \leq Z \leq z]$ for different value of z) as discussed in Unit 14 of MST-003.

For example, if test is right-tailed and calculated value of test statistic Z is 1.23 then

$$\text{p-value} = P[Z \geq z] = P[Z \geq 1.23] = 0.5 - P[0 < Z < 1.23]$$

$$= 0.5 - 0.3907 \left[\begin{array}{l}\text{From Z-table given in Appendix}\\ \text{of Block 1of this course.}\end{array}\right]$$

$$= 0.1093$$

Now, you can try the following exercises.

E1) If an investigator observed that the calculated value of test statistic lies in non-rejection region then he/she will

(i) reject the null hypothesis

(ii) accept the null hypothesis

(iii) not reject the null hypothesis

Write the correct option.

E2) If we have null and alternative hypotheses as

$H_0: \theta = \theta_0$ and $H_1: \theta \neq \theta_0$

then the rejection (critical) region lies in

(i) left tail

(ii) right tail

(iii) both tails

Write the correct option.

E3) If test is two-tailed and calculated value of test statistic Z is 2.42 then calculate the p-value for the Z-test.

10.3 TESTING OF HYPOTHESIS FOR POPULATION MEAN USING Z-TEST

In previous section, we have discussed the general procedure for Z-test. Now we are discussing the Z-test for testing the hypothesis or claim about the population mean when sample size is large. Let population under study has mean μ and variance σ^2, where μ is unknown and σ^2 may be known or unknown. We will consider both cases under this heading. For testing a hypothesis about population mean we draw a random sample $X_1, X_2, \ldots, X_n$ of size $n > 30$ from this population. As we know that for drawing the inference about the population mean we generally use sample mean and for test statistic, we require the mean and standard error of sampling distribution of the statistic (mean). Here, we are considering large sample so we know by central limit theorem that sample mean is asymptotically normally distributed with mean μ and variance σ^2/n whether parent population is **normal or non-normal**. That is, if $\bar{X}$ is the sample mean of the random sample then

$$E(\bar{X}) = \mu, \; Var(\bar{X}) = \frac{\sigma^2}{n} \qquad \ldots (1)$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore \quad SE(\bar{X}) = \sqrt{Var(\bar{X})} = \frac{\sigma}{\sqrt{n}} \qquad \ldots (2)$$

Now, follow the same procedure as we have discussed in previous section, that is, first of all we have to setup null and alternative hypotheses. Since here we want to test the hypothesis about the population mean so we can take the null and alternative hypotheses as

$$H_0 : \mu = \mu_0 \text{ and } H_1 : \mu \neq \mu_0 \quad [\text{for two-tailed test}] \quad \begin{bmatrix} \text{Here, } \theta = \mu \text{ and } \theta_0 = \mu_0 \\ \text{if we compare it with} \\ \text{general procedure.} \end{bmatrix}$$

or

$$\left.\begin{array}{l} H_0 : \mu \leq \mu_0 \text{ and } H_1 : \mu > \mu_0 \\ H_0 : \mu \geq \mu_0 \text{ and } H_1 : \mu < \mu_0 \end{array}\right\} [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - E(\bar{X})}{SE(\bar{X})}$$

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \qquad \begin{bmatrix} \text{Using equations (1) and (2) and} \\ \text{under } H_0 \text{ we assume that } \mu = \mu_0. \end{bmatrix}$$

When we assume that the null hypothesis is true then we are actually assuming that the population parameter is equal to the value in the null hypothesis. For example, we assume that $\mu = 60$ whether the null hypothesis is $\mu = 60$ or $\mu \leq 60$ or $\mu \geq 60$.

The sampling distribution of the test statistic depends upon σ^2 that it is known or unknown. Therefore, two cases arise:

Case I: When σ^2 is known

In this case, the test statistic follows the normal distribution with mean 0 and variance unity when the sample size is the large as the population under study is normal or non-normal. If the sample size is small then test statistic Z follows the normal distribution only when population under study is normal. Thus,

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0, 1)$$

Case II: When σ^2 is unknown

In this case, we estimate σ^2 by the value of sample variance (S^2) where,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

Then become test statistic $\frac{\bar{X} - \mu_0}{S / \sqrt{n}}$ follows the t-distribution with (n−1) df as the sample size is large or small provided the population under study follows normal as we have discussed in Unit 2 of this course. But when population under study is not normal and sample size is large then this test statistic approximately follows normal distribution with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \mu_0}{S / \sqrt{n}} \sim N(0,1)$$

After that, we calculate the value of test statistic as may be the case (σ^2 is known or unknown) and compare it with the critical value given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in the previous section.

From above discussion of testing of hypothesis about population mean, we note following point:

(i) When σ^2 is known then we apply the Z-test as the population under study is normal or non-normal for the large sample. But when sample size is small then we apply the Z-test only when population under study is normal.

(ii) When σ^2 is unknown then we apply the t-test only when the population under study is normal as sample size is large or small. But when the assumption of normality is not fulfilled and sample size is large then we can apply the Z-test.

(iii) When sample is small and σ^2 is known or unknown and the form of the population is not known then we apply the non-parametric test as we will be discussed in Block 4 of this course.

Following examples will help you to understand the procedure more clearly.

Example 1: A light bulb company claims that the 100-watt light bulb it sells has an average life of 1200 hours with a standard deviation of 100 hours. For testing the claim 50 new bulbs were selected randomly and allowed to burn out. The average lifetime of these bulbs was found to be 1180 hours. Is the company's claim is true at 5% level of significance?

Solution: Here, we are given that

Specified value of population mean = μ_0 = 1200 hours,

Population standard deviation = σ = 100 hours,

Sample size = n = 50

Sample mean = $\bar{X}$ = 1180 hours.

In this example, the population parameter being tested is population mean i.e. average life of a bulb (μ) and we want to test the company's claim that average life of a bulb is 1200 hours. So our claim is μ = 1200 and its complement is $\mu \neq 1200$. Since claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. So

$$H_0 : \mu = \mu_0 = 1200 \; [\text{average life of a bulb is 1200 hours}]$$

$$H_1 : \mu \neq 1200 \qquad [\text{average life of a bulb is not1200 hours}]$$

Also the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis regarding mean when population SD (variance) is known and sample size n = 50(> 30) is large. So we will go for Z-test.

Thus, for testing the null hypothesis the test statistic is given by

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

$$= \frac{1180 - 1200}{100 / \sqrt{50}} = \frac{-20}{14.14} = -1.41$$

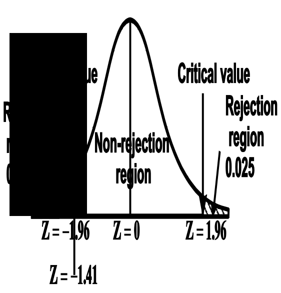

Fig. 10.4

The critical (tabulated) values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of test statistic Z (= –1.41) is greater than critical value (= − 1.96) and less than the critical value (= 1.96), that means it lies in non-rejection region as shown in Fig. 10.4, so we do not reject the null hypothesis. Since the null hypothesis is the claim so we support the claim at 5% level of significance.

Decision according to p-value:

The test is two-tailed, therefore,

p-value = $2P\left[Z \ge |z|\right] = 2P\left[Z \ge 1.41\right]$

$$= 2\left[0.5 - P\left[0 \le Z \le 1.41\right]\right] = 2\left(0.5 - 0.4207\right) = 0.1586$$

Since p-value (= 0.1586) is greater than α (= 0.05) so we do not reject the null hypothesis at 5% level of significance.

Decision according to confidence interval:

Here, test is two-tailed, therefore, we contract two-sided confidence interval for population mean.

Since population standard deviation is known, therefore, we can use $(1-\alpha)$ 100 % confidence interval for population mean when population variance is known which is given by

$$\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

Here, $\alpha = 0.05$, we have $z_{\alpha/2} = z_{0.025} = 1.96.$

Thus, 95% confidence interval for average life of a bulb is given by

$$\left[1180 - 1.96\frac{100}{\sqrt{50}}, 1180 + 1.96\frac{100}{\sqrt{50}}\right]$$

or $\quad \left[1180 - 27.71, 1180 + 27.71\right]$

or $\quad \left[1152.29, 1207.71\right]$

Since 95% confidence interval for average life of a bulb contains the value of the parameter specified by the null hypothesis, that is, $\mu = \mu_0 = 1200$ so we do not reject the null hypothesis.

Thus, we conclude that sample does not provide us sufficient evidence against the claim so we may assume that the company's claim that the average life of a bulb is 1200 hours is true.

Note 2: Here, we note that the decisions about null hypothesis based on three approaches (critical value or classical, p-value and confidence interval) are same. The learners are advised to make the decision about the claim or statement by using only one of the three approaches in the examination. Here, we used all these approaches only to give you an idea how they can be used in a given problem. Those learners who will opt biostatistics specialisation will see and realize the importance of confidence interval approach in Unit 16 of MSTE-004.

Example 2: A manufacturer of ball point pens claims that a certain pen manufactured by him has a mean writing-life at least 460 A-4 size pages. A purchasing agent selects a sample of 100 pens and put them on the test. The mean writing-life of the sample found 453 A-4 size pages with standard deviation 25 A-4 size pages. Should the purchasing agent reject the manufacturer's claim at 1% level of significance?

Solution: Here, we are given that

Specified value of population mean = μ_0 = 460, Sample size = n = 100,

Sample mean = $\bar{X}$ = 453,

Sample standard deviation = S = 25

Here, we want to test the manufacturer's claim that the mean writing-life (μ) of pen is at least 460 A-4 size pages. So our claim is $\mu \geq 460$ and its complement is $\mu < 460$. Since claim contains the equality sign so we can take the claim as the null hypothesis and the complement as the alternative hypothesis. So

$$H_0 : \mu \geq \mu_0 = 460 \text{ and } H_1 : \mu < 460$$

Also the alternative hypothesis is left-tailed so the test is left-tailed test.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. So we should used t-test for if writing-life of pen follows normal distribution. But it is not the case. Since sample size is n = 100 (n > 30) large so we go for Z-test. The test statistic of Z-test is given by

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

$$= \frac{453 - 460}{25/\sqrt{100}} = \frac{-7}{2.5} = -2.8$$

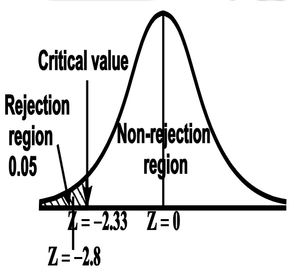

Fig. 10.5

We have from Table 10.1 the critical (tabulated) value for left-tailed Z test at 1% level of significance is $z_\alpha = -2.33$.

Since calculated value of test statistic Z (= –2.8) is less than the critical value (= −2.33), that means calculated value of test statistic Z lies in rejection region as shown in Fig. 10.5, so we reject the null hypothesis. Since the null hypothesis is the claim so we reject the manufacturer's claim at 1% level of significance.

Decision according to p-value:

The test is left-tailed, therefore,

$$\text{p-value} = P[Z \leq z] = P[Z \leq -2.8] = P[Z \geq 2.8] \quad \left[\begin{array}{l}\because Z \text{ is symetrical} \\ \text{about } Z = 0 \text{ line}\end{array}\right]$$

$$= 0.5 - P[0 \leq Z \leq 2.8] = 0.5 - 0.4974 = 0.0026$$

Since p-value (= 0.0026) is less than α (= 0.01) so we reject the null hypothesis at 1% level of significance.

Therefore, we conclude that the sample provide us sufficient evidence against the claim so the purchasing agent rejects the manufacturer's claim at 1% level of significance.

Now, you can try the following exercises.

E4) A sample of 900 bolts has a mean length 3.4 cm. Is the sample regarded to be taken from a large population of bolts with mean length 3.25 cm and standard deviation 2.61 cm at 5% level of significance?

E5) A big company uses thousands of CFL lights every year. The brand that the company has been using in the past has average life of 1200 hours. A new brand is offered to the company at a price lower than they are paying for the old brand. Consequently, a sample of 100 CFL light of new brand is tested which yields an average life of 1220 hours with standard deviation 90 hours. Should the company accept the new brand at 5% level of significance?

10.4 TESTING OF HYPOTHESIS FOR DIFFERENCE OF TWO POPULATION MEANS USING Z-TEST

In previous section, we have learnt about the testing of hypothesis about the population mean. But there are so many situations where we want to test the hypothesis about difference of two population means or two population means. For example, two manufacturing companies of bulbs are produced same type of bulbs and one may be interested to test that one is better than the other, an investigator may want to test the equality of the average incomes of the peoples living in two cities, etc. Therefore, we require an appropriate test for testing the hypothesis about the difference of two population means.

Let there be two populations, say, population-I and population-II under study. Also let μ_1, μ_2 and σ_1^2, σ_2^2 denote the means and variances of population-I and population-II respectively where both μ_1 and μ_2 are unknown but σ_1^2 and σ_2^2 may be known or unknown. We will consider all possible cases here. For testing the hypothesis about the difference of two population means, we draw a random sample of large size n_1 from population-I and a random sample of large size n_2 from population-II. Let $\bar{X}$ and $\bar{Y}$ be the means of the samples selected from population-I and II respectively.

These two populations may or may not be normal but according to the central limit theorem, the sampling distribution of difference of two large sample means asymptotically normally distributed with mean $(\mu_1 - \mu_2)$ and variance $(\sigma_1^2/n_1 + \sigma_2^2/n_2)$ as described in Unit 2 of this course.

Thus,

$$E\left(\bar{X}-\bar{Y}\right) = E\left(\bar{X}\right) - E\left(\bar{Y}\right) = \mu_1 - \mu_2 \qquad \ldots (3)$$

and

$$\mathrm{Var}\left(\bar{X}-\bar{Y}\right) = \mathrm{Var}\left(\bar{X}\right) + \mathrm{Var}\left(\bar{Y}\right) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore \quad SE\left(\bar{X}-\bar{Y}\right) = \sqrt{\mathrm{Var}\left(\bar{X}-\bar{Y}\right)} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad \ldots (4)$$

Now, follow the same procedure as we have discussed in Section 10.2, that is, first of all we have to setup null and alternative hypotheses. Here, we want to test the hypothesis about the difference of two population means so we can take the null hypothesis as

$$H_0 : \mu_1 = \mu_2 \text{ (no difference in means)} \quad \begin{bmatrix} \text{Here, } \theta_1 = \mu_1 \text{ and } \theta_2 = \mu_2 \\ \text{if we compare it with} \\ \text{general procedure.} \end{bmatrix}$$

or $H_0: \mu_1 - \mu_2 = 0$ (difference in two means is 0)

and the alternative hypothesis as

$H_1: \mu_1 \neq \mu_2$ [for two-tailed test]

or $\left.\begin{array}{l} H_0: \mu_1 \leq \mu_2 \text{ and } H_1: \mu_1 > \mu_2 \\ H_0: \mu_1 \geq \mu_2 \text{ and } H_1: \mu_1 < \mu_2 \end{array}\right\}$ [for one-tailed test]

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\left(\bar{X} - \bar{Y}\right) - E\left(\bar{X} - \bar{Y}\right)}{SE\left(\bar{X} - \bar{Y}\right)}$$

or $$Z = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad [\text{using equations (3) and (4)}]$$

Since under null hypothesis we assume that $\mu_1 = \mu_2$, therefore, we have

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Now, the sampling distribution of the test statistic depends upon σ_1^2 and σ_2^2 that both are known or unknown. Therefore, four cases arise:

Case I: When $\sigma_1^2 \,\&\, \sigma_2^2$ are known and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

In this case, the test statistic follows normal distribution with mean 0 and variance unity when the sample sizes are large as both the populations under study are normal or non-normal. But when sample sizes are small then test statistic Z follows normal distribution only when populations under study are normal, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Case II: When $\sigma_1^2 \,\&\, \sigma_2^2$ are known and $\sigma_1^2 \neq \sigma_2^2$

In this case, the test statistic also follows the normal distribution as described in case I, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

Case III: When $\sigma_1^2 \,\&\, \sigma_2^2$ are unknown and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

In this case, σ^2 is estimated by value of pooled sample variance S_p^2 where,

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[n_1 S_1^2 + n_2 S_2^2\right]$$

and

$$S_1^2 = \frac{1}{(n_1 - 1)} \sum_{i=1}^{n_1} \left(X_i - \bar{X} \right)^2 \text{ and } S_2^2 = \frac{1}{(n_2 - 1)} \sum_{i=1}^{n_2} \left(Y_i - \bar{Y} \right)^2$$

and test statistic follows t-distribution with ($n_1 + n_2 - 2$) degrees of freedom as the sample sizes are large or small provided populations under study follow normal distribution as described in Unit 2 of this course. But when the populations are under study are not normal and sample sizes n_1 and $n_2$are large (> 30) then by central limit theorem, test statistic approximately normally distributed with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Case IV: When $\sigma_1^2 \& \sigma_2^2$ are unknown and $\sigma_1^2 \neq \sigma_2^2$

In this case, $\sigma_1^2 \& \sigma_2^2$ are estimated by the values of the sample variances $S_1^2 \& S_2^2$ respectively and the exact distribution of test statistic is difficult to derive. But when sample sizes n_1 and n_2 are large (> 30) then central limit theorem, the test statistic approximately normally distributed with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim N(0, 1)$$

After that, we calculate the value of test statistic and compare it with the critical value given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in Section10.2.

From above discussion of testing of hypothesis about population mean, we note following point:

(i) When $\sigma_1^2 \& \sigma_2^2$ are known then we apply the Z-test as both the population under study are normal or non-normal for the large sample. But when sample sizes are small then we apply the Z-test only when populations under study are normal.

(ii) When $\sigma_1^2 \& \sigma_2^2$ are unknown then we apply the t-test only when the populations under study are normal as sample sizes are large or small. But when the assumption of normality is not fulfilled and sample sizes are large then we can apply the Z-test.

(iii) When samples are small and $\sigma_1^2 \& \sigma_2^2$ are known or unknown and the form of the population is not known then we apply the non-parametric test as we will be discussed in Block 4 of this course.

Let us do some examples based on above test.

Example 3: In two samples of women from Punjab and Tamilnadu, the mean height of 1000 and 2000 women are 67.6 and 68.0 inches respectively. If population standard deviation of Punjab and Tamilnadu are same and equal to 5.5 inches then, can the mean heights of Punjab and Tamilnadu women be regarded as same at 1% level of significance?

Solution: We are given

$$n_1 = 1000,\ n_2 = 2000,\ \bar{X} = 67.6,\ \bar{Y} = 68.0 \text{ and } \sigma_1 = \sigma_2 = \sigma = 5.5$$

Here, we wish to test that the mean height of Punjab and Tamilnadu women is same. If μ_1 and μ_2 denote the mean heights of Punjab and Tamilnadu women respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis regarding two population means. The standard deviations of both populations are known and sample sizes are large, so we should go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{67.6 - 68.0}{\sqrt{\left(\frac{(5.5)^2}{1000} + \frac{(5.5)^2}{2000}\right)}} = \frac{-0.4}{5.5\sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{-0.4}{5.5 \times 0.0387} = -1.88$$

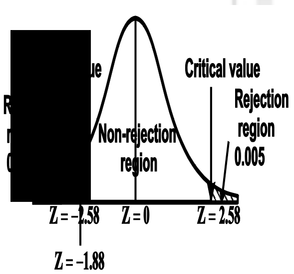

Fig. 10.6

The critical (tabulated) values for two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since calculated value of Z ($= -1.88$) is greater than the critical value ($= -2.58$) and less than the critical value ($= 2.58$), that means it lies in non-rejection region as shown in Fig. 10.6, so we do not reject the null hypothesis i.e. we fail to reject the claim.

Decision according to p-value:

The test is two-tailed, therefore,

$$\text{p-value} = 2P\left[Z \geq |z|\right] = 2P\left[Z \geq 1.88\right]$$

$$= 2\left[0.5 - P\left[0 \leq Z \leq 1.88\right]\right] = 2\left(0.5 - 0.4699\right) = 0.0602$$

Since p-value (= 0.0602) is greater than $\alpha (= 0.01)$ so we do not reject the null hypothesis at1% level of significance.

Thus, we conclude that the samples do not provide us sufficient evidence against the claim so we may assume that the average height of women of Punjab and Tamilnadu is same.

Example 4: A university conducts both face to face and distance mode classes for a particular course indented both to be identical. A sample of 50 students of face to face mode yields examination results mean and SD respectively as:

$$\bar{X} = 80.4, \quad S_1 = 12.8$$

and other sample of 100 distance-mode students yields mean and SD of their examination results in the same course respectively as:

$$\bar{Y} = 74.3, \quad S_2 = 20.5$$

Are both educational methods statistically equal at 5% level?

Solution: Here, we are given that

$$n_1 = 50, \quad \bar{X} = 80.4, \quad S_1 = 12.8;$$

$$n_2 = 100, \quad \bar{Y} = 74.3, \quad S_2 = 20.5$$

We wish to test that both educational methods are statistically equal. If μ_1 and μ_2 denote the average marks of face to face and distance mode students respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

We want to test the null hypothesis regarding two population means when standard deviations of both populations are unknown. So we should go for t-test if population of difference is known to be normal. But it is not the case. Since sample sizes are large (n_1, and $n_2 > 30$) so we go for Z-test.

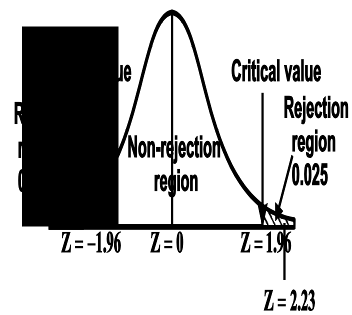

Fig. 10.7

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$= \frac{80.4 - 74.3}{\sqrt{\frac{(12.8)^2}{50} + \frac{(20.5)^2}{100}}} = \frac{6.1}{\sqrt{3.28 + 4.20}} = 2.23$$

The critical (tabulated) values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= 2.23) is greater than the critical values (= ±1.96), that means it lies in rejection region as shown in Fig. 10.7, so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Decision according to p-value:

The test is two-tailed, therefore,

$$\text{p-value} = 2P\left[Z \geq |z|\right] = 2P\left[Z \geq 2.23\right]$$

$$= 2\left[0.5 - P\left[0 \leq Z \leq 2.23\right]\right] = 2\left(0.5 - 0.4871\right) = 0.0258$$

Since p-value (= 0.0258) is less than $\alpha (= 0.05)$ so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so both methods of education, i.e. face-to-face and distance-mode, are not statistically equal.

Now, you can try the following exercises.

E6) Two brands of electric bulbs are quoted at the same price. A buyer was tested a random sample of 200 bulbs of each brand and found the following information:

	Mean Life (hrs.)	SD(hrs.)
Brand A	1300	41
Brand B	1280	46

Is there significant difference in the mean duration of their lives of two brands of electric bulbs at 1% level of significance?

E7) Two research laboratories have identically produced drugs that provide relief to BP patients. The first drug was tested on a group of 50 BP patients and produced an average 8.3 hours of relief with a standard deviation of 1.2 hours. The second drug was tested on 100 patients, producing an average of 8.0 hours of relief with a standard deviation of 1.5 hours. Does the first drug provide a significant longer period of relief at a significant level of 5%?

10.5 TESTING OF HYPOTHESIS FOR POPULATION PROPORTION USING Z-TEST

In Section 10.3, we have discussed the procedure of testing of hypothesis for population mean when sample size is large. But in many real world situations, in business and other areas where collected data are in form of counts or the collected data are classified into two categories or groups according to an attribute or a characteristic. For example, the peoples living in a colony may be classified into two groups (male and female) with respect to the characteristic sex, the patients in a hospital may be classified into two groups as cancer and non-cancer patients, the lot of articles may be classified as defective and non-defective, etc. Here, collected data are available in dichotomous or binary outcomes which is a special case of nominal scale and the data categorized into two mutually exclusive and exhaustive classes generally known as success and failure out comes. For example, the characteristic sex can be measured as success if male and failure if female or vice versa. So in such situations, proportion is suitable measure to apply.

In such situations, we require a test for testing a hypothesis about population proportion.

For this purpose, let $X_1, X_2, ..., X_n$ be a random sample of size n taken from a population with population proportion P. Also let X denotes the number of observations or elements possess a certain attribute (number of successes) out of n observations of the sample then sample proportion p can be defined as

$$p = \frac{X}{n} \leq 1$$

As we have seen in Section 2.4 of the Unit 2 of this course that mean and variance of the sampling distribution of sample proportion are

$$E(p) = P \text{ and } Var(p) = \frac{PQ}{n}$$

where, Q = 1– P.

Now, two cases arise:

Case I: When sample size is not sufficiently large i.e. either of the conditions np > 5 or nq > 5 does not meet, then we use exact binomial test. But exact binomial test is beyond the scope of this course.

Case II: When sample size is sufficiently large, such that np > 5 and nq > 5 then by central limit theorem, the sampling distribution of sample proportion p is approximately normally distributed with mean and variance as

$$E(p) = P \text{ and } Var(p) = \frac{PQ}{n} \qquad \dots (5)$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore \qquad SE(p) = \sqrt{\frac{PQ}{n}} \qquad \dots (6)$$

Now, follow the same procedure as we have discussed in Section 10.2, first of all we setup null and alternative hypotheses. Since here we want to test the hypothesis about specified value P_0 of the population proportion so we can take the null and alternative hypotheses as

$$H_0 : P = P_0 \text{ and } H_1 : P \neq P_0 \text{ [for two-tailed test]} \begin{bmatrix}\text{Here, } \theta = P \text{ and } \theta_0 = P_0 \\ \text{if we compare it with} \\ \text{general procedure.}\end{bmatrix}$$

or

$$\left.\begin{array}{l} H_0 : P \leq P_0 \text{ and } H_1 : P > P_0 \\ H_0 : P \geq P_0 \text{ and } H_1 : P < P_0 \end{array}\right\} \quad \text{[for one-tailed test]}$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - E(p)}{SE(p)}$$

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} \sim N(0, 1) \text{ under } H_0 \text{ [using equations (5) and (6)]}$$

After that, we calculate the value of test statistic and compare it with the critical value(s) given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in Section 10.2.

Let us do some examples of testing of hypothesis about population proportion.

Example 5: A machine produces a large number of items out of which 25% are found to be defective. To check this, company manager takes a random sample of 100 items and found 35 items defective. Is there an evidence of more deterioration of quality at 5% level of significance?

Solution: The company manager wants to check that his machine produces 25% defective items. Here, attribute under study is defectiveness. And we define our success and failure as getting a defective or non defective item.

Let P = Population proportion of defectives items = 0.25(= P_0)

p = Observed proportion of defectives items in the sample = 35/100 = 0.35

Here, we want to test that machine produces more defective items, that is, the proportion of defective items (P) greater than 0.25. So our claim is P > 0.25

and its complement is $P \le 0.25$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. So

$$H_0 : P \le P_0 = 0.25 \text{ and } H_1 : P > 0.25$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad np = 100 \times 0.35 = 35 > 5$$

$$nq = 100 \times (1 - 0.35) = 100 \times 0.65 = 65 > 5$$

We see that condition of normality meets, so we can go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}}$$

$$= \frac{0.35 - 0.25}{\sqrt{\frac{0.25 \times 0.75}{100}}} = \frac{0.10}{0.0433} = 2.31$$

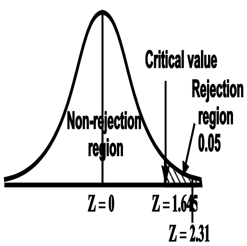

Fig. 10.8

Since test is right-tailed so the critical value at 5% level of significance is $z_\alpha = z_{0.05} = 1.645$.

Since calculated value of test statistic Z (= 2.31) is greater than the critical value (= 1.645), that means it lies in the rejection region as shown in Fig. 10.8. So we reject the null hypothesis and support the alternative hypothesis i.e. we support the claim at 5% level of significance.

Decision according to p-value:

The test is right-tailed, therefore,

$$\text{p-value} = P[Z \ge z] = P[Z \ge 2.31]$$

$$= 0.5 - P[0 < Z < 2.31] = 0.5 - 0.4896$$

$$= 0.0104$$

Since p-value (= 0.0104) is less than $\alpha\,(= 0.05)$ so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the claim so we may assume that deterioration in quality exists at 5% level of significance.

Example 6: A die is thrown 9000 times and draw of 2 or 5 is observed 3100 times. Can we regard that die is unbiased at 5% level of significance.

Solution: Let getting a 2 or 5 be our success, and getting a number other than 2 or 5 be a failure then in usual notions, we have

$$n = 9000,\ X = \text{number of successes} = 3100,\ p = 3100/9000 = 0.3444$$

Here, we want to test that the die is unbiased and we know that if die is unbiased then proportion or probability of getting 2 or 5 is

P = Probability of getting a 2 or 5

= Probability of getting 2 + Probability of getting 5

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0.3333$$

So our claim is P = 0.3333 and its complement is P ≠ 0.3333. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : P = P_0 = 0.3333 \text{ and } H_1 : P \neq 0.3333$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad np = 9000 \times 0.3444 = 3099.6 > 5$$

$$nq = 9000 \times (1 - 0.3444) = 9000 \times 0.6556 = 5900.4 > 5$$

We see that condition of normality meets, so we can go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}}$$

$$= \frac{0.3444 - 0.3333}{\sqrt{\frac{0.3333 \times 0.6667}{9000}}} = \frac{0.0111}{0.005} = 2.22$$

Since test is two-tailed so the critical values at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= 2.22) is greater than the critical value (= 1.96), that means it lies in rejection region, so we reject the null hypothesis i.e. we reject our claim.

Decision according to p-value:

Since test is two-tailed so

$$\text{p-value} = 2P\left[Z \geq |z|\right] = 2P\left[Z \geq 2.22\right]$$

$$= 2\left[0.5 - P\left[0 < Z < 2.22\right]\right] = 2\left(0.5 - 0.4868\right) = 0.0264$$

Since p-value (= 0.0264) is less than $\alpha (= 0.05)$, so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that the sample provides us sufficient evidence against the claim so die cannot be considered as unbiased.

Now, you can try the following exercises.

E8) In a sample of 100 MSc. Economics first year students of a University, it was seen that 54 students came from Science background and the rest are from other background. Can we assume that 50% of the students are from Science background in MSc. Economics first year students in the University at 1% level of significance?

E9) Out of 200 patients who are given a particular injection 180 survived. Test the hypothesis that the survival rate is more than 80% at 5% level of significance?

10.6 TESTING OF HYPOTHESIS FOR DIFFERENCE OF TWO POPULATION PROPORTIONS USING Z-TEST

In Section 10.5, we have discussed the testing of hypothesis about the population proportion. In some cases, we are interested to test the hypothesis about difference of two population proportions of an attributes in the two different populations or groups. For example, one may wish to test whether the proportions of alcohol drinkers in the two cities are same, one may wish to test proportion of literates in a group of people is greater than the proportion of literates in other group of people, etc. Therefore, we require the test for testing the hypothesis about the difference of two population proportions.

Let there be two populations, say, population-I and population-II under study. And also let we draw a random sample of size n_1 from population-I with population proportion P_1 and a random sample of size n_2 from population-II with population proportion P_2. If X_1 and X_2 are the number of observations /individuals / items / units possessing the given attribute in the sample of sizes n_1 and n_2 respectively then sample proportions can be defined as

$$p_1 = \frac{X_1}{n_1} \text{ and } p_2 = \frac{X_2}{n_2}$$

As we have seen in Section 2.5 of the Unit 2 of this course that mean and variance of the sampling distribution of difference of sample proportions are

$$E(p_1 - p_2) = P_1 - P_2$$

and variance

$$Var(p_1 - p_2) = \frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}$$

where, $Q_1 = 1 – P_1$ and $Q_2 = 1 – P_2$.

Now, two cases arise:

Case I: When sample sizes are not sufficiently large i.e. any of the conditions $n_1p_1 > 5$ or $n_1q_1 > 5$ or $n_2p_2 > 5$ or $n_2q_2 > 5$ does not meet, then we use exact binomial test. But exact binomial test is beyond the scope of this course.

Case II: When sample sizes are sufficiently large, such that $n_1p_1 > 5$, $n_1q_1 > 5$, $n_2p_2 > 5$ and $n_2q_2 > 5$ then by central limit theorem, the sampling distribution of sample proportions p_1 and p_2 are approximately normally as

$$p_1 \sim N\left(P_1, \frac{P_1Q_1}{n_1}\right) \text{ and } p_2 \sim N\left(P_2, \frac{P_2Q_2}{n_2}\right)$$

Also, by the property of normal distribution described in Unit 13 of MST-003, the sampling distribution of the difference of sample proportions follows normal distribution with mean

$$E(p_1 - p_2) = E(p_1) - E(p_2) = P_1 - P_2 \qquad \ldots (7)$$

and variance

$$Var(p_1 - p_2) = Var(p_1) + Var(p_2) = \frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}$$

That is,

$$p_1 - p_2 \sim N\left(P_1 - P_2, \frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}\right)$$

Thus, standard error is given by

$$SE(p_1 - p_2) = \sqrt{Var(p_1 - p_2)} = \sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}} \qquad \ldots (8)$$

Now, follow the same procedure as we have discussed in Section 10.2, first of all we have to setup null and alternative hypotheses. Here, we want to test the hypothesis about the difference of two population proportions so we can take the null hypothesis as

$$H_0: P_1 = P_2 \text{ (no difference in proportions)} \left[\begin{array}{l}\text{Here, } \theta_1 = P_1 \text{ and} \\ \theta_2 = P_2 \text{ if we compare} \\ \text{it with general} \\ \text{procedure.}\end{array}\right]$$

or $\quad H_0: P_1 - P_2 = 0$ (difference in two proportions is 0)

and the alternative hypothesis may be

$$H_1: P_1 \neq P_2 \qquad [\text{for two-tailed test}]$$

or $\left.\begin{array}{l} H_0: P_1 \le P_2 \text{ and } H_1: P_1 > P_2 \\ H_0: P_1 \ge P_2 \text{ and } H_1: P_1 < P_2 \end{array}\right\} \quad [\text{for one-tailed test}]$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{(p_1 - p_2) - E(p_1 - p_2)}{SE(p_1 - p_2)}$$

or $$Z = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}} \qquad [\text{using equations (7) and (8)}]$$

Since under null hypothesis we assume that $P_1 = P_2 = P$, therefore, we have

$$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where, Q = 1 - P.

Generally, P is unknown then it is estimated by the value of pooled proportion $\hat{P}$, where

$$\hat{P} = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} \text{ and } \hat{Q} = 1 - \hat{P}$$

After that, we calculate the value of test statistic and compare it with the critical value(s) given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in Section 10.2.

Now, it is time for doing some examples for testing of hypothesis about the difference of two population proportions.

Example 7: In a random sample of 100 persons from town A, 60 are found to be high consumers of wheat. In another sample of 80 persons from town B, 40 are found to be high consumers of wheat. Do these data reveal a significant difference between the proportions of high wheat consumers in town A and town B (at α = 0.05)?

Solution: Here, attribute under study is high consuming of wheat. And we define our success and failure as getting a person of high consumer of wheat and not high consumer of wheat respectively.

We are given that

n_1 = total number of persons in the sample of town A = 100

n_2 = total number of persons in the sample of town B = 80

X_1 = number of persons of high consumer of wheat in town A = 60

X_2 = number of persons of high consumer of wheat in town B = 40

The sample proportion of high wheat consumers in town A is

$$p_1 = \frac{X_1}{n_1} = \frac{60}{100} = 0.60$$

and the sample proportion of wheat consumers in town B is

$$p_2 = \frac{X_2}{n_2} = \frac{40}{80} = 0.50$$

Here, we want to test that the proportion of high consumers of wheat in two towns, say, P_1 and P_2, is not same. So our claim is $P_1 \neq P_2$ and its complement is $P_1 = P_2$. Since the complement contains the equality sign, so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : P_1 = P_2 = P \text{ and } H_1 : P_1 \neq P_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Before proceeding further, first we have to check whether the condition of normality meets or not.

$\therefore \qquad n_1p_1 = 100 \times 0.60 = 60 > 5,\ n_1q_1 = 100 \times 0.40 = 40 > 5$

$n_2p_2 = 80 \times 0.50 = 40 > 5,\ n_2q_2 = 80 \times 0.50 = 40 > 5$

We see that condition of normality meets, so we can go for Z-test.

The estimate of the combined proportion (P) of high wheat consumers in two towns is given by

$$\hat{P} = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{60 + 40}{100 + 80} = \frac{5}{9}$$

$$\hat{Q} = 1 - \hat{P} = 1 - \frac{5}{9} = \frac{4}{9}$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.60 - 0.50}{\sqrt{\frac{5}{9} \times \frac{4}{9}\left(\frac{1}{100} + \frac{1}{80}\right)}} = \frac{0.10}{0.0745} = 1.34$$

The critical values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (=1.34) is less than the critical value (= 1.96) and greater than critical value (= −1.96), that means calculated value of Z lies in non-rejection region, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim.

Decision according to p-value:

Since the test is two-tailed, therefore

$$\text{p-value} = 2\,P\left[Z \ge z\right] = 2P\left[Z \ge 1.34\right]$$

$$= 2\left[0.5 - P\left[0 \le Z \le 1.34\right]\right] = 2\left(0.5 - 0.4099\right) = 0.1802$$

Since p-value (= 0.1802) is greater than $\alpha (= 0.05)$ so we do not reject the null hypothesis at 5% level of significance.

Thus, we conclude that the samples provide us the sufficient evidence against the claim so we may assume that the proportion of high consumers of wheat in two towns A and B is same.

Example 8: A machine produced 60 defective articles in a batch of 400. After overhauling it produced 30 defective in a batch of 300. Has the machine improved due to overhauling? (Take $\alpha = 0.01$).

Solution: Here, the machine produced articles and attribute under study is defectiveness. And we define our success and failure as getting a defective or non defective article. Therefore, we are given that

X_1 = number of defective articles produced by the machine before overhauling = 60

X_2 = number of defective articles produced by the machine after overhauling = 30 and $n_1 = 400$, $n_2 = 300$,

Let p_1 = Observed proportion of defective articles in the sample before the overhauling

$$= \frac{X_1}{n_1} = \frac{60}{400} = 0.15$$

and p_2 = Observed proportion of defective articles in the sample after the overhauling

$$= \frac{X_2}{n_2} = \frac{30}{300} = 0.10$$

Here, we want to test that machine improved due to overhauling that means the proportion of defective articles is less after overhauling. If P_1 and P_2 denote the proportion defectives before and after the overhauling the machine so our claim is $P_1 > P_2$ and its complement $P_1 \le P_2$. Since the complement contains the equality sign so we can take the complement as the null hypothesis and claim as the alternative hypothesis. Thus,

$$H_0 : P_1 \le P_2 \text{ and } H_1 : P_1 > P_2$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Since P is unknown, so the pooled estimate of proportion is given by

$$\hat{P} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{60+30}{400+300} = \frac{90}{700} = \frac{9}{70} \text{ and } \hat{Q} = 1 - \hat{P} = 1 - \frac{9}{70} = \frac{61}{70}.$$

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad n_1p_1 = 400 \times 0.15 = 60 > 5,\ n_1q_1 = 400 \times 0.85 = 340 > 5$$

$$n_2p_2 = 300 \times 0.10 = 30 > 5,\ n_2q_2 = 300 \times 0.90 = 270 > 5$$

We see that condition of normality meets, so we can go for Z-test.

For testing the null hypothesis, the statistic is given by

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.15 - 0.10}{\sqrt{\frac{9}{70} \times \frac{61}{70}\left(\frac{1}{400} + \frac{1}{300}\right)}} = \frac{0.05}{0.0256} = 1.95$$

The critical value for right-tailed test at 1% level of significance is $z_\alpha = z_{0.01} = 2.33$.

Since calculated value of Z (= 1.95) is less than the critical value (= 2.33) that means calculated value of Z lies in non-rejection region, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 1% level of significance.

Decision according to p-value:

Since the test is right-tailed, therefore,

p-value = $P[Z \ge z] = P[Z \ge 1.95]$

$$= 0.5 - P\left[0 \le Z \le 1.95\right] = 0.5 - 0.4744 = 0.0256$$

Since p-value (= 0.0256) is greater than $\alpha(= 0.01)$ so we do not reject the null hypothesis at1% level of significance.

Thus, we conclude that the samples provide us sufficient evidence against the claim so the machine has not been improved after overhauling.

Now, you can try the following exercises.

E10) The proportions of literates between groups of people of two districts A and B are tested. Out of the 100 persons selected at random from each district, 50 from district A and 40 from district B are found literates. Test whether the proportion of literate persons in two districts A and B is same at 1% level of significance?

E11) In a large population 30% of a random sample of 1200 persons had blue-eyes and 20% of a random sample of 900 persons had the same blue-eyes in another population. Test the proportion of blue-eyes persons is same in two populations at 5% level of significance.

10.7 TESTING OF HYPOTHESIS FOR POPULATION VARIANCE USING Z-TEST

In Section 10.3, we have discussed testing of hypothesis for population mean but when analysing quantitative data, it is often important to draw conclusion about the average as well as the variability of a characteristic of under study. For example, a company manufactured the electric bulbs and the manager of the company would probably be interested in determining the average life of the bulbs and also determining whether or not the variability in the life of bulbs is within acceptable limits, the product controller of a milk company may be interested to know variance of the amount of fat in the whole milk processed by the company is no more than the specified level, etc. So we require a test for this purpose.

The procedure of testing a hypothesis for population variance or standard deviation is similar to the testing of population mean.

For testing a hypothesis about the population variance, we draw a random sample $X_1, X_2, ..., X_n$ of size $n > 30$ from the population with mean μ and variance σ^2 where, μ be known or unknown.

We know that by central limit theorem that sample variance is asymptotically normally distributed with mean σ^2 and variance $2\sigma^4/n$ whether parent population is **normal or non-normal**. That is, if S^2 is the sample variance of the random sample then

$$E(S^2) = \sigma^2 \text{ and } Var(S^2) = \frac{2\sigma^4}{n} \qquad \dots (9)$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore \qquad SE\left(S^2\right) = \sqrt{Var\left(S^2\right)} = \sqrt{\frac{2}{n}}\sigma^2 \qquad \dots (10)$$

The general procedure of this test is explained in the next page.

As we are doing so far in all tests, first Step in hypothesis testing problems is to setup null and alternative hypotheses. Here, we want to test the hypothesis specified value σ_0^2 of the population variance σ^2 so we can take our null and alternative hypotheses as

$$H_0 : \sigma^2 = \sigma_0^2 \text{ and } H_1 : \sigma^2 \neq \sigma_0^2 \qquad [\text{for two-tailed test}]$$

or

$$\left.\begin{array}{l} H_0 : \sigma^2 \leq \sigma_0^2 \text{ and } H_1 : \sigma^2 > \sigma_0^2 \\ H_0 : \sigma^2 \geq \sigma_0^2 \text{ and } H_1 : \sigma^2 < \sigma_0^2 \end{array}\right\} \qquad [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{S^2 - E(S^2)}{SE(S^2)} \sim N(0,1)$$

$$Z = \frac{S^2 - \sigma_0^2}{\sigma_0^2\sqrt{\frac{2}{n}}} \qquad \left[\begin{array}{l}\text{Using equations (9) and (10) and} \\ \text{under } H_0 : \sigma^2 = \sigma_0^2\end{array}\right]$$

After that, we calculate the value of test statistic and compare it with the critical value given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in Section 10.2.

Note 2: When population under study is normal then for testing the hypothesis about population variance or population standard deviation we use chi-square test which will be discussed in Unit 12 of this course. Whereas when the distribution of the population under study is not known and sample size is large then we apply Z-test as discussed above.

Now, it is time to do example based on above test.

Example 9: A random sample of size 65 screws is taken from a population of big box of screws and measured their length (in mm) which gives sample variance 9.0. Test that the two years old population variance 10.5 is still maintained at present at 5% level of significance.

Solution: We are given that

$$n = 65, \qquad S^2 = 9.0, \qquad \sigma_0^2 = 10.5$$

Here, we want to test that the two years old screw length population variance (σ^2) is still maintained at 10.5. So our claim is $\sigma^2 = 10.5$ and its complement is $\sigma^2 \neq 10.5$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \sigma^2 = \sigma_0^2 = 10.5 \text{ and } H_1 : \sigma^2 \neq 10.5$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, the distribution of population under study is not known and sample size is large ($n > 30$) so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{S^2 - \sigma_0^2}{\sigma_0^2\sqrt{\frac{2}{n}}} \sim N(0,1)$$

$$= \frac{9.0 - 10.5}{10.5\sqrt{\frac{2}{65}}} = \frac{-1.5}{10.5 \times 0.175} = \frac{-1.5}{1.84} = -0.81$$

The critical values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= −0.81) is less than critical value (= 1.96) and greater than the critical value (= −1.96), that means it lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim.

Thus, we conclude that sample fails to provide us sufficient evidence against the claim so we may assume that two years old screw length population variance is still maintained at 10.5mm.

Now, you can try the following exercise.

E12) A random sample of size 120 bulbs is taken from a lot which gives the standard deviation of the life of electric bulbs 7 hours. Test the standard deviation of the life of bulbs of the lot is 6 hours at 5% level of significance.

10.8 TESTING OF HYPOTHESIS FOR TWO POPULATION VARIANCES USING Z-TEST

In previous section, we have discussed testing of hypothesis about the population variance. But there are so many situations where we want to test the hypothesis about equality of two population variances or standard deviations. For example, an economist may want to test whether the variability in incomes differ in two populations, a quality controller may want to test whether the quality of the product is changing over time, etc.

Let there be two populations, say, population-I and population-II under study. Also let μ_1, μ_2 and σ_1^2, σ_2^2 denote the means and variances of population-I and population-II respectively where both σ_1^2 and σ_2^2 are unknown but μ_1 and μ_2 may be known or unknown. For testing the hypothesis about equality of two population variances or standard deviations, we draw a random sample of large size n_1 from population-I and a random sample of large size n_2 from population-II. Let S_1^2 and S_2^2 be the sample variances of the samples selected from population-I and population-II respectively.

These two populations **may or may not be normal** but according to the central limit theorem, the sampling distribution of difference of two large sample variances asymptotically normally distributed with mean $(\sigma_1^2 + \sigma_2^2)$ and variance $(2\sigma_1^4/n_1 + 2\sigma_2^4/n_2)$.

Thus,

$$E\left(S_1^2 - S_2^2\right) = E\left(S_1^2\right) - E\left(S_2^2\right) = \sigma_1^2 - \sigma_2^2 \qquad \ldots (11)$$

and

$$\text{Var}\left(S_1^2 - S_2^2\right) = \text{Var}\left(S_1^2\right) + \text{Var}\left(S_2^2\right) = \frac{2\sigma_1^4}{n_1} + \frac{2\sigma_2^4}{n_2}$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore \quad SE\left(S_1^2 - S_2^2\right) = \sqrt{Var\left(S_1^2 - S_2^2\right)} = \sqrt{\frac{2\sigma_1^4}{n_1} + \frac{2\sigma_2^4}{n_2}} \qquad \ldots (12)$$

Now, follow the same procedure as we have discussed in Section 10.2, that is, first of all we have to setup null and alternative hypothesis. Here, we want to test the hypothesis about the two population variances, so we can take our null and alternative hypotheses as

$$H_0 : \sigma_1^2 = \sigma_2^2 = \sigma^2 \text{ and } H_1 : \sigma_1^2 \neq \sigma_2^2 \qquad [\text{for two-tailed test}]$$

or

$$\left.\begin{aligned} &H_0 : \sigma_1^2 \leq \sigma_2^2 \text{ and } H_1 : \sigma_1^2 > \sigma_2^2 \\ &H_0 : \sigma_1^2 \geq \sigma_2^2 \text{ and } H_1 : \sigma_1^2 < \sigma_2^2 \end{aligned}\right\} \qquad [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\left(S_1^2 - S_2^2\right) - E\left(S_1^2 - S_2^2\right)}{SE\left(S_1^2 - S_2^2\right)} \sim N(0,1)$$

or

$$Z = \frac{\left(S_1^2 - S_2^2\right) - \left(\sigma_1^2 - \sigma_2^2\right)}{\sqrt{\frac{2\sigma_1^4}{n_1} + \frac{2\sigma_2^4}{n_2}}} \quad [\text{using equations (11) and (12)}]$$

Since under null hypothesis we assume that $\sigma_1^2 = \sigma_2^2 = \sigma^2$, therefore, we have

$$Z = \frac{S_1^2 - S_2^2}{\sigma^2 \sqrt{2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0,1)$$

Generally, population variances σ_1^2 and σ_2^2 are unknown, so we estimate them by their corresponding sample variances S_1^2 and S_2^2 as

$$\hat{\sigma}_1^2 = S_1^2 \quad \text{and} \quad \hat{\sigma}_2^2 = S_2^2$$

Thus, the test statistic Z is given by

$$Z = \frac{S_1^2 - S_2^2}{\sqrt{\left(\frac{2S_1^4}{n_1} + \frac{2S_2^4}{n_2}\right)}} \sim N(0,1)$$

After that, we calculate the value of test statistic as may be the case and compare it with the critical value given in **Table 10.1** at prefixed level of significance α. Take the decision about the null hypothesis as described in Section 10.2.

Note 3: When populations under study are normal then for testing the hypothesis about equality of population variances we use F- test which will be discussed in Unit 12 of this course. Whereas when the form of the populations under study is not known and sample sizes are large then we apply Z-test as discussed above.

Now, it is time to do an example based on above test.

Example 10: A comparative study of variation in weights (in pound) of Army-soldiers and Navy- sailors was made. The sample variance of the weight of 120 soldiers was 60 pound^2 and the sample variance of the weight of 160 sailors was 70 pound^2. Test whether the soldiers and sailors have equal variation in their weights. Use 5% level of significance.

Solution: Given that

$$n_1 = 120,\ S_1^2 = 60,\ n_2 = 160,\ S_2^2 = 70$$

We want to test that the Army-soldiers and Navy-sailors have equal variation in their weights. If σ_1^2 and σ_2^2 denote the variances in the weight of Army-soldiers and Navy-sailors so our claim is $\sigma_1^2 = \sigma_2^2$ and its complement is $\sigma_1^2 \neq \sigma_2^2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \begin{bmatrix} \text{Army-soldiers and Navy-sailors} \\ \text{have equal variation in their weights} \end{bmatrix}$$

and the alternative hypothesis as

$$H_1 : \sigma_1^2 \neq \sigma_2^2 \quad \begin{bmatrix} \text{Army-soldiers and Navy-sailors} \\ \text{have different variation in their weights} \end{bmatrix}$$

Here, the distributions of populations under study are not known and sample sizes are large $(n_1 = 120 > 30,\ n_2 = 160 > 30)$ so we can go for Z-test.

Since population variances are unknown so for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{S_1^2 - S_2^2}{\sqrt{\left(\frac{2S_1^4}{n_1} + \frac{2S_2^4}{n_2}\right)}}$$

$$= \frac{60 - 70}{\sqrt{\frac{2 \times (60)^2}{120} + \frac{2 \times (70)^2}{160}}}$$

$$= \frac{-10}{\sqrt{60.0 + 61.25}} \quad \frac{-10}{11.01} = -0.91$$

The critical values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= −0.91) is less than critical value (= 1.96) and greater than the critical value (= −1.96), that means it lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that the Army-soldiers and Navy-sailors have equal variation in their weights.

Now, you can try the following exercise.

E 13) Two sources of raw materials of bulbs are under consideration by a bulb manufacturing company. Both sources seem to have similar characteristics but the company is not sure about their respective uniformity. A sample of 52 lots from source A yields variance 25 and a sample of 40 lots from source B yields variance of 12. Test whether the variance of source A significantly differs to the variances of source B at $\alpha = 0.05$?

We now end this unit by giving a summary of what we have covered in it.

10.9 SUMMARY

In this unit we have covered the following points:

1. How to judge a given situation whether we should go for large sample test or not.
2. Applying the Z-test for testing the hypothesis about the population mean and difference of two population means.
3. Applying the Z-test for testing the hypothesis about the population proportion and difference of two population proportions.
4. Applying the Z-test for testing the hypothesis about the population variance and two population variances.

10.10 SOLUTIONS / ANSWERS

E1) Since we have a rule that if the observed value of test statistic lies in rejection region then we reject the null hypothesis and if calculated value of test statistic lies in non-rejection region then we do not reject the null hypothesis. Therefore in our case, we do not reject the null hypothesis. So (iii) is the correct answer. Remember always on the basis of the one sample we never accept the null hypothesis.

E2) Since the test is two-tailed therefore rejection region will lie under both tails.

E3) Since test is two-tailed, therefore,

p-value = $2P\left[Z \geq |z|\right] = 2P\left[Z \geq 2.42\right]$

$$= 2\left[0.5 - P\left[0 \leq Z \leq 2.42\right]\right] = 2\left(0.5 - 0.4922\right) = 0.0156$$

E4) We are given that

$$n = 900,\ \bar{X} = 3.4 \text{ cm},\ \mu_0 = 3.25 \text{ cm and } \sigma = 2.61 \text{ cm}$$

Here, we wish to test that the sample comes from a large population of bolts with mean (μ) 3.25cm. So our claim is $\mu = 3.25$cm and its complement is $\mu \neq 3.25$cm. Since the claim contains the equality sign so we can take the claim as the null hypothesis and the complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 3.25 \text{ and } H_1 : \mu \neq 3.25$$

Since the alternative hypothesis is two-tailed, so the test is two-tailed test.

Here, we want to test the hypothesis regarding population mean when population SD is unknown, so we should use t-test if the population of bolts known to be normal. But it is not the case. Since the sample size is large (n > 30) so we can go for Z-test instead of t-test as an approximate. So test statistic is given by

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

$$= \frac{3.40 - 3.25}{2.61/\sqrt{900}} = \frac{0.15}{0.087} = 1.72$$

The critical (tabulated) values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of test statistic Z (= 1.72) is less than the critical value (= 1.96) and greater than critical value (= −1.96), that means it lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim at 5% level of significance.

Thus, we conclude that sample does not provide us sufficient evidence against the claim so we may assume that the sample comes from the population of bolts with mean 3.25cm.

E5) Here, we given that

$$\mu_0 = 1200,\ n = 100,\ \bar{X} = 1220,\ S = 90$$

Here, the company may accept the new CFL light when average life of CFL light is greater than 1200 hours. So the company wants to test that the new brand CFL light has an average life greater than 1200 hours. So our claim is $\mu > 1200$ and its complement is $\mu \le 1200$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu \le \mu_0 = 1200 \text{ and } H_1 : \mu > 1200$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Here, we want to test the hypothesis regarding population mean when population SD is unknown, so we should use t-test if the distribution of life of bulbs known to be normal. But it is not the case. Since the sample size is large (n > 30) so we can go for Z-test instead of t-test. Therefore, test statistic is given by

$$Z = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}$$

$$= \frac{1220 - 1200}{90/\sqrt{100}} = \frac{20}{9} = 2.22$$

The critical values for right-tailed test at 5% level of significance is $z_\alpha = z_{0.05} = 1.645$.

Since calculated value of test statistic Z (= 2.22) is greater than critical value (= 1.645), that means it lies in rejection region so we reject the null hypothesis and support the alternative hypothesis i.e. we support our claim at 5% level of significance.

Thus, we conclude that sample does not provide us sufficient evidence against the claim so we may assume that the company accepts the new brand of bulbs.

E6) Given that

$$n_1 = 200, \quad \bar{X} = 1300, \quad S_1 = 41;$$

$$n_2 = 200, \quad \bar{Y} = 1280, \quad S_2 = 46$$

Here, we want to test that there is significant difference in the mean duration of their lives of two brands of electric bulbs. If μ_1 and μ_2 denote the mean lives of two brands of electric bulbs respectively then our claim is $\mu_1 \neq \mu_2$ and its complement is $\mu_1 = \mu_2$. Since the complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

We want to test the null hypothesis regarding equality of two population means. The standard deviations of both populations are unknown so we should go for t-test if population of difference is known to be normal. But it is not the case. Since sample sizes are large (n_1, and $n_2 > 30$) so we go for Z-test.

So for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$= \frac{1300 - 1280}{\sqrt{\frac{(41)^2}{200} + \frac{(46)^2}{200}}} = \frac{20}{\sqrt{8.41 + 10.58}} = \frac{20}{4.36} = 4.59$$

The critical (tabulated) values for two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since calculated value of test statistic Z (= 4.59) is greater than the critical values (= ± 2.58), that means it lies in rejection region, so we reject the null hypothesis an support the alternative hypothesis i.e. support the claim at 1% level of significance.

Thus, we conclude that samples do not provide us sufficient evidence against the claim so there is significant difference in the mean duration of their lives of two brands of electric bulbs.

E7) Given that

$$n_1 = 50, \quad \bar{X} = 8.3, \quad S_1 = 1.2;$$

$$n_2 = 100, \quad \bar{Y} = 8.0, \quad S_2 = 1.5$$

Here, we want to test that the first drug provides a significant longer period of relief than the other. If μ_1 and μ_2 denote the mean relief time due to first and second drugs respectively then our claim is $\mu_1 > \mu_2$ and its complement is $\mu_1 \le \mu_2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 \le \mu_2 \text{ and } H_1 : \mu_1 > \mu_2$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

We want to test the null hypothesis regarding equality of two population means. The standard deviations of both populations are unknown. So we should go for t-test if population of difference is known to be normal. But it is not the case. Since sample sizes are large (n_1, and $n_2 > 30$) so we go for Z-test.

So for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$= \frac{8.3 - 8.0}{\sqrt{\frac{(1.2)^2}{50} + \frac{(1.5)^2}{100}}} = \frac{0.3}{\sqrt{0.0288 + 0.0255}}$$

$$= \frac{0.3}{0.2265} = 1.32$$

The critical (tabulated) value for right-tailed test at 5% level of significance is $z_\alpha = z_{0.05} = 1.645$.

Since calculated value of test statistic Z (= 1.32) is less than the critical value (=1.645), that means it lies in non-rejection region, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so the first drug does not has longer period of relief than the other.

E8) Here, students are classified as Science background and other. We define our success and failure as getting student of Science background and other respectively. We are given that

n = Total number of students is the sample = 100

X = Number of students from Science background = 54

p = Sample proportion of students from Science background

$$= \frac{54}{100} = 0.54$$

We want to test whether 50% of the students are from Science background in MSc. If P denotes the proportion of first year Science background students in the University. So our claim is $P = P_0 = 0.5$ and its complement is $P \ne 0.5$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$H_0 = P = P_0 = 0.5$ (= 50%)

H_1: $P \neq 0.5$ [Science background differs to 50%]

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad np = 100 \times 0.54 = 54 > 5$$

$$nq = 100 \times (1 - 0.54) = 100 \times 0.46 = 46 > 5$$

We see that condition of normality meets, so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} = \frac{0.54 - 0.50}{\sqrt{0.5 \times 0.5/100}} \qquad [\because Q_0 = 1 - P_0]$$

$$= \frac{0.04}{0.05} = 0.80$$

The critical (tabulated) values for two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since calculated value of test statistic Z (= 0.80) is less than the critical value (= 2.58) and greater than critical value (= −2.58), that means it lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that sample fails to provide us sufficient evidence against the claim so we may assume that 50% of the first year students in MSc. Economics in the University are from Science background.

E9) We define our success and failure as a patient is survived and not survived. Here, we are given that

n = 200

X = Number of survived patients who are given a particular injection

= 180

p = Sample proportion of survived patients who are given a particular injection

$$= \frac{X}{n} = \frac{180}{200} = 0.9$$

$$P_0 = 80\% = \frac{80}{100} = 0.80 \Rightarrow Q_0 = 1 - P_0 = \frac{80}{100} = 0.20$$

Here, we want to test that the survival rate of the patients is more than 80%. If P denotes the proportion of survival patients then our claim is $P > 0.80$ and its complement is $P \leq 0.80$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : P \leq P_0 = 0.80 \text{ and } H_1 : P > 0.80$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad np = 200 \times 0.9 = 180 > 5$$

$$nq = 200 \times (1 - 0.9) = 200 \times 0.1 = 20 > 5$$

We see that condition of normality meets, so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}}$$

$$= \frac{0.9 - 0.8}{\sqrt{\frac{(0.8)(0.2)}{200}}} = \frac{0.1}{0.0283} = 3.53$$

The critical (tabulated) value for right-tailed test at 5% level of significance is $z_{\alpha} = z_{0.05} = 1.645$.

Since calculated value of test statistic Z (= 3.53) is greater than the critical value (=1.645), that means it lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. support our claim at 5% level of significance.

Thus, we conclude that sample fails to provide us sufficient evidence against the claim so we may assume that the survival rate is greater than 80% in the population using that injection.

E10) Let X_1 and X_2 stand for number of literates in districts A and B respectively. Therefore, we are given that

$$n_1 = 100, X_1 = 50 \Rightarrow p_1 = \frac{X_1}{n_1} = \frac{50}{100} = 0.50$$

$$n_2 = 100, X_2 = 40 \Rightarrow p_2 = \frac{X_2}{n_2} = \frac{40}{100} = 0.40$$

Here, we want to test whether the proportion of literate persons in two districts A and B is same. If P_1 and P_2 denote the proportions of literate persons in two districts A and B respectively then our claim is $P_1 = P_2$ and its complement $P_1 \neq P_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : P_1 = P_2 = P \text{ and } H_1 : P_1 \neq P_2$$

Since the alternative hypothesis is two-tailed so the test is two tailed test.

The estimate of the combined proportion (P) of literates in districts A and B is given by

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{50 + 40}{100 + 100} = 0.45$$

$$\hat{Q} = 1 - \hat{P} = 1 - 0.45 = 0.55$$

Before proceeding further, first we have to check whether the condition of normality meets or not.

$$\therefore \quad n_1p_1 = 100\times 0.50 = 50 > 5,\ n_1q_1 = 100\times 0.50 = 50 > 5$$

$$n_2p_2 = 100\times 0.40 = 40 > 5,\ n_2q_2 = 100\times 0.60 = 60 > 5$$

We see that condition of normality meets, so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$$

$$= \frac{0.50 - 0.40}{\sqrt{0.45\times 0.55\left(\frac{1}{100}+\frac{1}{100}\right)}} \quad = \frac{0.10}{0.0704} = 1.42$$

The critical (tabulated) values for two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since calculated value of test statistic Z (= 1.42) is less than the critical value (= 2.58) and greater than critical value (= −2.58), that means it lies in non-rejection region, so we do not reject the null hypothesis i.e. we support our claim at 1% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that the proportion of literates in districts A and B is equal.

E11) Here, we are given that

$$n_1 = 1200, \quad p_1 = 30\% = 0.30$$

$$n_2 = 900, \quad p_2 = 20\% = 0.20$$

Here, we want to test that the proportion of blue-eye persons in both the population is same. If P_1 and P_2 denote the proportions of blue-eye persons in two populations respectively then our claim is $P_1 = P_2$ and its complement $P_1 \neq P_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : P_1 = P_2 = P \text{ and } H_1 : P_1 \neq P_2$$

Since the alternative hypothesis is two-tailed so the test is two tailed test.

The estimate of the combined proportion (P) of literates in districts A and B is given by

$$\hat{P} = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{1200\times 0.30 + 900\times 0.20}{1200 + 900} = 0.257$$

$$\hat{Q} = 1 - \hat{P} = 1 - 0.257 = 0.743$$

Before proceeding further, first we have to check whether the condition of normality meets or not.

$\therefore \quad n_1p_1 = 1200 \times 0.30 = 360 > 5,\ n_1q_1 = 1200 \times 0.70 = 840 > 5$

$n_2p_2 = 900 \times 0.20 = 180 > 5,\ n_2q_2 = 900 \times 0.80 = 720 > 5$

We see that condition of normality meets, so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.30 - 0.20}{\sqrt{0.257 \times 0.743\left(\frac{1}{1200} + \frac{1}{900}\right)}} = \frac{0.10}{0.019} = 5.26$$

The critical (tabulated) values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 1.96$.

Since calculated value of test statistic Z (= 5.26) is greater than critical values (= ±1.96), that means it lies in rejection region, so we reject the null hypothesis i.e. we reject our claim at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so the proportion of blue-eyed person in both the population is not same.

E12) Here, we are given that

$$n = 120, \quad S = 7, \quad \sigma_0 = 6$$

Here, we want to test that standard deviation (σ) of the life of bulbs of the lot is 6 hours. So our claim is σ = 6 and its complement is σ ≠ 6. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \sigma = \sigma_0 = 6 \text{ and } H_1 : \sigma \neq 6$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, the distribution of population under study is not known and sample size is large (n > 30) so we can go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{S^2 - \sigma_0^2}{\sigma_0^2\sqrt{\frac{2}{n}}} \sim N(0,1)$$

$$= \frac{(7)^2 - (6)^2}{(6)^2\sqrt{\frac{2}{120}}} = \frac{13}{36 \times 0.129} = \frac{13}{4.64} = 2.80$$

The critical values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= 2.8) is greater than critical values (= ±1.96), that means it lies in rejection region, so we reject the null hypothesis i.e. we reject our claim at 5% level of significance.

Thus, we conclude that sample provides us sufficient evidence against the claim so standard deviation of the life of bulbs of the lot is not 6.0 hours.

E13) Here, we are given that

$$n_1 = 52, \quad S_1^2 = 25$$

$$n_2 = 40, \quad S_2^2 = 12$$

Here, we want to test that variance of source A significantly differs to the variances of source B. If σ_1^2 and σ_2^2 denote the variances in the raw materials of sources A and B respectively so our claim is $\sigma_1^2 \neq \sigma_2^2$ and its complement is $\sigma_1^2 = \sigma_2^2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ and } H_1 : \sigma_1^2 \neq \sigma_2^2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, the distributions of populations under study are not known and sample sizes are large $(n_1 = 52 > 30,\ n_2 = 40 > 30)$ so we can go for Z-test.

Since population variances are unknown so for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{S_1^2 - S_2^2}{\sqrt{\left(\frac{2S_1^4}{n_1} + \frac{2S_2^4}{n_2}\right)}}$$

$$= \frac{25-12}{\sqrt{\frac{2(25)^2}{52} + \frac{2(12)^2}{40}}} = \frac{13}{5.5} = 2.36$$

The critical values for two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since calculated value of Z (= 2.36) is greater than critical values (= ±1.96), that means it lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. we support our claim at 5% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so variance of source A significantly differs to the variance of source B.